ON AN EFFECTIVE METHOD OF SOLUTION OF CERTAIN INTEGRAL EQUATIONS OF THE THEORY OF ELASTICITY AND MATHEMATICAL PHYSICS

(OB ODNOM EFFEKTIVNOM METODE RESHENIIA NEKOTORYKH INTEGRAL'NYKH URAVNENII TEORII UPRUGOSTI I MATEMATICHESKOI FIZIKI)

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In paper [1] the plane problem of the indenting of an elastic layer by a rigid die by a flat base is solved.

The solution is obtained in the form of a series which converges in some interval $0 \leqslant \lambda < c \neq \infty$ ($\lambda = h / a$ is the relative thickness of the layer). However, this method is not applicable if $z_k - z_l \neq z_m$ (k, l and m are any integers : this is in the notation of [2]). Moreover, the characteristic singularities which occur at the points where the boundary conditions change are not separated out in the solution. Also, the convergence of the series is not established in the whole interval $0 \leq \lambda < \infty$.

Another approach to the investigation of this kind of problem is proposed below, based on a study of the corresponding integral equations [2]. The solution is obtained in the form of a series which converges in the entire interval $0 \leq \lambda < \infty$.

An approximate method is presented which makes it possible to write the solution out in a form which is convenient for practical use. An example is given.

1. We first give an algorithm which permits us to construct the solution of an infinite series of linear algebraic equations under certain conditions.

We shall investigate the system [2]

$$[A + B(a)]X = D \tag{1.1}$$

where A and $B(a)$ are infinite matrices, X and D are infinite dimensional vectors, or, equivalently, infinite sequences : A^{-1} is the inverse of the matrix A.

We shall consider as proved the existence of a unique solution of the system (1.1) belonging to a subspace c_1 of the space m [3], when $D \in s_1$.

Here c_1 consists of infinite sequences such that

$$\sum_{l=1}^{\infty} |x_l| \, l^{-1} < \infty \tag{1.2}$$

and s_1 is such that if $Y \in s_1$, then $AA^{-1} \cdot Y$ is associative.

We shall assume the following properties (*) of the infinite matrices:

*) These conditions are sufficient for quasi-regularity of the corresponding system represented in canonical form.

1) The elements of the matrix $B(\alpha)$ are functions which may be continued analytically in the complex plane, are regular in the region $\mathrm{Re}\, z > 0$ and all approach zero as $\mathrm{Re}\, z \to \infty$;

$$\text{2)} \qquad A^{-1}B(z) \ (c_1 \to c_1)$$

$$\text{3)} \qquad B(z)\, X \in s_1 \quad \text{for} \quad X \in c_1 \ (\beta < \mathrm{Re}\, z < \infty) \tag{1.3}$$

$$\text{4)} \qquad \|A^{-1}B(z)\|_m < \infty$$

5) For some $D \in s_1$ the expression $A^{-1}D = X_0 \in c_1$.

We shall denote an infinite matrix all elements of which are zero except the elements of the first n columns by $B_n(z)$.

Lemma. The solution of the system

$$[A + B_n(z)]\, X_n = D \qquad (\mathrm{Re}\, z > \beta) \tag{1.4}$$

is determined by the recurrence relation

$$X_n = (x_l(n)) = \left(x_l(n-1) - x_n(n-1)\frac{\varepsilon_{l,n}(n-1)}{1+\varepsilon_{n,n}(n-1)}\right) \qquad X_0 = (x_l(0))$$

$$A^{-1} = (\tau_{l,k}(0)) \tag{1.5}$$

The following notation is used in (1.5)

$$[A + B_k(z)]^{-1}[B_n(z) - B_{n-1}(z)] = (\varepsilon_{l,n}(k)) \tag{1.6}$$

$$[A + B_k(z)]^{-1} = \left(\tau_{l,m}(k-1) - \tau_{k,m}(k-1)\frac{\varepsilon_{l,n}(k-1)}{1+\varepsilon_{k,k}(k-1)}\right) \tag{1.7}$$

A solution does not exist here (is unbounded with respect to the norm of m) on the countable set of zeros of the function $1 + \varepsilon_{n,n}(n-1)$.

In order to prove this, we premultiply Equation (1.4) by A^{-1} (by virtue of the assumptions (2), (3) and (5) this can be done) and we rewrite (1.4) in the form

$$X_n = X_0 - A^{-1}B_n(z)\, X_n \tag{1.8}$$

Setting $n = 1$ and taking $\mathrm{Re}\, z$ sufficiently large, we can find, in view of the assumptions (1) and (4), a z_0 such that

$$\|A^{-1}B_1(z)\|_m \leqslant q < 1 \qquad (\mathrm{Re}\, z \geqslant \mathrm{Re}\, z_0) \tag{1.9}$$

The method of successive approximations can then be applied to Equation (1.8), which permits us to write the solution in the form

$$X_1 = \left[I + \sum_{k=1}^{\infty} (-1)^k (A^{-1}B_1(z))^k\right] X_0 \tag{1.10}$$

Using the notation (1.6) and (1.7), we obtain (1.5) with $n = 1$ for $\mathrm{Re}\, z \geq \mathrm{Re}\, z_0$.

It is easy to show that the analytic continuation of the function X_1 in the region $\beta \leq \mathrm{Re}\, z < \mathrm{Re}\, z_0$ carried out with the aid of (1.5) also satisfies Equation (1.4) for all $\mathrm{Re}\, z \geq \beta$, except for the points which are the zeros of the function $1 + \varepsilon_{1,1}(0)$.

To construct X_2, we rewrite (1.4) for $n = 2$ in the form

$$[A + B_1 + B_2 - B_1]\, X_2 = D \tag{1.11}$$

If the matrix $[A + B_1]^{-1}$ is known, then the conditions of the first part of the proof are satisfied. To obtain this matrix, it is necessary to solve Equation

$$[A + B_1]\, Y = I \quad (I \text{ is the identity matrix})$$

The solution of this has the form (1, 7) for $k = 1$, which is obtained in just the same way as X_1. Thus, X_2 can be written in accordance with (1, 5) for $n = 2$.

Repeating the process n times, we obtain X_n which, as can be verified, may be unbounded at the points indicated in the lemma.

Theorem. If (2) to (4) are satisfied uniformly with respect to z in the region $0 < \beta \leq \mathrm{Re}\, z < \infty$, then $\| X - X_n \| \to \infty$ as $n \to \infty$.

Here $X = X(z)$ is the solution of Equation (1, 1) when $B = B(z)$.

The solution obtained may be unbounded at the points satisfying Equation

$$\lim_{n\to\infty} [1 + \varepsilon_{n,n}(n-1)] = 0 \tag{1.12}$$

By virtue of condition (4), an N can be found so that for $n > N$

$$\| A^{-1}(B - B_n) \|_m \leqslant q < 1 \; (\beta \leqslant \mathrm{Re}\, z \leqslant \infty) \tag{1.13}$$

It is then possible to apply the method of successive approximations to Equation

$$(A + B - B_n) Z_n = D$$

and to write its solution $Z_n \in c_1$ in the form

$$Z_n = \left\{ I + \sum_{k=1}^{\infty} (-1)^k [A^{-1}(B - B_n)]^k \right\} X_0 \qquad (\beta \leqslant \mathrm{Re}\, z < \infty) \tag{1.14}$$

$$\| Z_n - X_0 \| \leqslant \frac{\| A^{-1}(B - B_n) \|}{1 - \| A^{-1}(B - B_n) \|} \| X_0 \| \tag{1.15}$$

The inverse of the matrix corresponding can be represented in the form

$$A_0^{-1} = (A + B - B_n)^{-1} = \left\{ I + \sum_{k=1}^{\infty} (-1)^k [A^{-1}(B - B_n)]^k \right\} A^{-1} \tag{1.16}$$

We now rewrite the original equation (1, 1) in the form

$$(A_0 + B_n(z)) X = D \qquad (A_0 = A + B(z) - B_n(z),\ n > N) \tag{1.17}$$

The lemma which has been proved above can be applied if the matrix A_0^{-1} of Equation (1, 16) is taken as A^{-1} and Z_n of (1, 14) is taken as X_0. It is easy to verify that the necessary conditions (1) to (5) are satisfied in this case as before.

Therefore, on the basis of (1, 14) and (1, 15), the solution of the original problem (1, 1) can be written out in the form $X = X(Z_n)$, which is bounded almost everywhere in the right half of the complex plane and is analytic there. Taking any other number $k > N$, we can find the solution $X = X(Z_k)$, where at all points of analyticity in z

$$X(Z_n) \equiv X(Z_k) \tag{1.18}$$

These two functions are analytic and coincide for $\mathrm{Re}\, z > \mathrm{Re}\, z^*$ [2]. But then, because of the arbitrariness of n in the relation (1, 18), we can pass to the limit $n \to \infty$ and here for fixed k the identity is not violated. Then, however, from Equations (1, 15) and (1, 16) it is evident that X_0 may be taken as Z_n and correspondingly A^{-1} as A_0^{-1}. This last result means that the theorem is proved.

Let us determine the error in the nth approximation. We take $n > N$ and find the solution X_n of Equation (1, 4). Then the solution of Equation (1, 1) can be written in a form analogous to (1, 14), viz.

$$X = \left\{ I + \sum_{k=1}^{\infty} (-1)^k [A^{-1}(B - B_n)]^k \right\} X_n \tag{1.19}$$

From this we obtain the estimate

$$\| X - X_n \| \leqslant \frac{\| A^{-1}(B - B_n) \|_m}{1 - \| A^{-1}(B - B_n) \|_m} \| X_n \| \tag{1.20}$$

Since the solution X_n is almost everywhere bounded as $n \to \infty$, it is clear that $\|X - X_n\| \to 0$ at the same rate as $\|A^{-1}(B - B_n)\|_m \to 0$.

By requiring that the argument z in the solution $X(z)$ be on the real axis, we obtain the solution of the system (1.1). It is only necessary to verify that the given a does not belong to the set defined by the condition (1.12).

2. We shall now reduce certain integral equations to infinite systems of linear, algebraic equations.

a) Let us examine Equation

$$\int_{-a}^{a} k(x - \xi) q_\eta(\xi) d\xi = \pi e^{i\eta x}, \qquad k(t) = \int_0^\infty \frac{L(u)}{u} \cos ut \, du \qquad (|x| \leqslant a) \tag{2.1}$$

where $u^{-1}L(u)$ is a meromorphic function whose properties are described in detail in [2]. A solution $q_\eta(\chi) \in L_1(-a, a)$ is sought. As is shown in [2], the solution of this integral equation has the form

$$q_\eta(x) = K^{-1}(\eta) e^{i\eta x} + \sum_{k=1}^{\infty} [y_k^+(a, \eta) \exp iz_k(a + x) + y_k^-(a, \eta) \exp iz_k(a - x)]$$

$$Y^+(y_k^+) \in c_1, \qquad Y^-(y_k^-) \in c_1 \tag{2.2}$$

The corresponding infinite system is representable in the form (1.1), where

$$A = (a_{r,l}) = \left(\frac{1}{\zeta_r - z_l}\right), \qquad B = (b_{r,l}) = \left(\pm \frac{\exp 2aiz_l}{\zeta_r + z_l}\right)$$

$$D = (d_r) = \left(K(\eta)\left(\frac{e^{-i\eta a}}{\eta - \zeta_r} \mp \frac{e^{i\eta a}}{\eta + \zeta_r}\right)\right) \tag{2.3}$$

$$y_k^+(a, \eta) = y_k^-(a, -\eta), \qquad x_l^\pm = y_l^+ \pm y_l^-, \qquad X = (x_l^\pm)$$

in which either the upper or lower signs must be taken throughout.

b) Let us consider Equations

$$\int_0^{2\pi} d\psi \int_0^a q(\rho, \psi) k(r, \rho, \varphi, \psi) \rho d\rho = 2\pi J_n(\eta r) \cos n\varphi \qquad (0 \leqslant \varphi < 2\pi, \ 0 \leqslant r \leqslant a) \tag{2.4}$$

$$k(r, \rho, \varphi, \psi) = \int_0^\infty L(u) J_0(uR/h) du, \qquad R = \sqrt{r^2 + \rho^2 + 2\rho r \cos(\varphi - \psi)} \tag{2.5}$$

Using the representation of the function $L(u)$ in the form of the sum of its principal parts and also applying the "addition formula" for Bessel functions, we may rewrite Equation (2.5) in the form (2.6)

$$k(r, \rho, \varphi, \psi) = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} 2i\gamma_m s_m [2 - \delta_{0,k}] \cos k(\varphi - \psi) \qquad \begin{pmatrix} I_k(\gamma_m \rho) K_k(\gamma_m r) \; r > \rho \\ I_k(\gamma_m r) K_k(\gamma_m \rho) \; r < \rho \end{pmatrix}$$

$$\delta_{i,k} = \begin{cases} 0 \ (i \neq k), \\ 1 \ (i = k), \end{cases} \qquad s_m = \left\{\left[\frac{\zeta_m}{L(\zeta_m)}\right]^1\right\}^{-1}, \qquad \zeta_m = i\gamma_m, \qquad z_m = i\delta_m$$

where $J_n(z)$ is the Bessel function, and $I_n(z)$ and $K_n(z)$ are the modified Bessel functions.

Just as in [2], it can be shown that the solution of Equation (2.6)

$$q(\rho, \psi) \in L_1(0, a; 0, 2\pi)$$

has the form

$$q(\rho, \psi) = [v_{n,0} J_n(\eta\rho) + \sum_{l=1}^{\infty} v_{n,l} K_n(\delta_l a) I_n(\delta_l \rho)] \cos n\psi \quad \begin{pmatrix} 0 \leqslant \rho \leqslant a \\ 0 \leqslant \psi < 2\pi \end{pmatrix} \tag{2.7}$$

where the $v_{n,k}$ are the elements of an infinite sequence $V_n \in c_1$.

In the same way as in [2], we obtain the following infinite system for the determination of the $v_{n,k}$:

$$\sum_{l=1}^{\infty} \frac{\delta_l K_n(\delta_l a) I_{n-1}(\delta_l a) K_n(\gamma_m a) + \gamma_m K_n(\delta_l a) I_n(\delta_l a) K_{n-1}(\gamma_m a)}{(\delta_l^2 - \gamma_m^2) K_n(\gamma_m a)} v_{n,l} =$$
$$= \frac{\eta K_n(\gamma_m a) J_{n-1}(\eta a) + \gamma_m K_{n-1}(\gamma_m a) J_n(\eta a)}{(\eta^2 + \gamma_m^2) K(\eta) K_n(\gamma_m a)} \qquad (m = 1, 2, \ldots) \tag{2.8}$$

We now examine the limiting values of the left-hand side of the system (2.8) as $a \to \infty$. Making use of the asymptotic formulas for the Bessel functions, we find, in the limit, Expressions of the form

$$\sum_{l=1}^{\infty} \frac{i}{2z_l} \frac{1}{\zeta_m - z_l} v_{n,l}^* \qquad (v_{n,l}^* = \lim_{a\to\infty} a^{-1} v_{n,l}) \tag{2.9}$$

The system (2.8) can then be represented in the form (1.1) if the following notation is introduced :

$$A = (a_{m,l}) = \left(\frac{1}{\zeta_m - z_l}\right), \qquad D = (d_m) = \left(i \frac{\eta K_n(\gamma_m a) J_{n-1}(\eta a) + \gamma_m K_{n-1}(\gamma_m a) J_n(\eta a)}{(\eta^2 + \gamma_m^2) K(\eta) K_n(\gamma_m a)}\right)$$

$$B = (b_{m,l}) =$$
$$= \left(\frac{\delta_l K_n(\delta_l a) I_{n-1}(\delta_l a) (K_n(\gamma_m a) + \gamma_m I_n(\delta_l (a) K_n(\delta_l a) K_{n-1}(\gamma_m a)}{(\delta_l^2 - \gamma_m^2) K_n(\gamma_m a)} 2a\delta_l - \frac{1}{\zeta_m - z_l}\right)$$
$$X = \left(\frac{i v_{n,l}}{2a z_l}\right) \tag{2.10}$$

It is obvious that by virtue of (2.9), and of the fact that the elements of the matrix $B(z)$ are analytic in the right half-plane, these elements vanish as $\operatorname{Re} z \to \infty$.

Let us find the expressions which are needed for the application of the results of Section 1 to the systems which have been obtained. In the case of Equation (2.1), we have [2]

$$x_l^{\pm}(0) = c_l(\eta) e^{-i\eta a} \pm c_l(-\eta) e^{i\eta a}, \qquad c_l(\eta) = [K_+(\eta)(\eta - z_l) K_+'(-z_l)]^{-1}$$

$$\varepsilon_{l,m}^{\pm}(0) = \sum_{r=1}^{\infty} \tau_{l,r}(0) b_{r,m} = \pm \frac{\exp 2aiz_m}{K_+'(-z_l)} \sum_{r=1}^{\infty} \frac{1}{[K_-^{-1}(\zeta_r)]'(\zeta_r - z_l)(\zeta_r + z_m)} =$$
$$= \pm \frac{\exp 2aiz_m K_+(z_m)}{K_+'(-z_l)(z_m + z_l)}, \qquad \tau_{l,r}(0) = \frac{1}{[K_-^{-1}(\zeta_r)]' K_+'(-z_l)(\zeta_r - z_l)} \tag{2.11}$$

The corresponding expressions for Equation (2.4) can be represented in the form

$$x_l(0) = \sum_{m=1}^{\infty} \tau_{l,m}(0) d_m = \frac{iJ_{n-1}(\eta a)[(\eta + z_l) K_-(\eta) + (\eta - z_l) K_+(\eta)]}{2K_+'(-z_l)(\eta^2 - z_l^2) K(\eta)} -$$
$$- \frac{iJ_n(\eta a)}{4K_+'(-z_l)} \left\{\left[\frac{K_{n-1}(-i\eta a)}{(\eta - z_l) K_n(-i\eta\alpha) K_+(\eta)} - \frac{K_{n-1}(i\eta a)}{(\eta + z_l) K_n(i\eta a) K_-(\eta)}\right] - \right.$$

$$-\text{v. p.}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{tK_{n-1}(-ita)K_-(t)\,dt}{(t-z_l)(t^2-\eta^2)K_n(-ita)}\Bigg\}$$

$$\varepsilon_{l,m}(0)=\sum_{k=1}^{\infty}\tau_{l,k}(0)\,b_{k,m}=\frac{iK_n(-iz_m a)I_{n-1}(-iz_m a)K_+(z_m)}{(z_m+z_l)K_+'(-z_l)}+$$

$$+\frac{I_n(-iz_m a)K_n(-iz_m a)}{2\pi i K_+'(-z_l)} \tag{2.12}$$

$$\left[\int_{-\infty}^{\infty}\frac{tK_-(t)K_{n-1}(-ita)\,dt}{(t-z_l)(t^2-z_m^2)K_n(-ita)}+2\pi i\,\frac{K_+(z_m)K_{n-1}(-iz_m a)}{2z_m(z_m+z_l)K_n(-iz_m a)}\right]-\delta_{l,m}$$

Now, in order to apply the results of Section 1 to the infinite systems (2. 3) and (2. 10), it is necessary to verify that the conditions (1) to (5) are satisfied . It is obvious that the condition (1) is satisfied in both cases . In order to verify that the conditions (2) to (5) are satisfied, we give the following estimates [2] which can be easily obtained from (2. 3) and (2. 11) :

$$A=(a_{r,l})\sim\left(\frac{1}{r-l}\right),\qquad B=(b_{r,l})\sim\left(\frac{e^{-lz}}{r+l}\right),\qquad D=(d_r)\sim\left(\frac{1}{r}\right) \tag{2.13}$$

$$X_0(x_l(0))\sim(l^{-1+\gamma}),\qquad A^{-1}=(\tau_{l,r})\sim\left(\frac{l^\gamma}{r^\gamma(r-l)}\right),\qquad (r\to\infty,\ l\to\infty,\ 0<\gamma<1)$$

The corresponding estimates for the case of the second system (2. 10) have the same form as (2. 13) for all the matrices except the matrix B. The following estimate is obtained for the matrix B :

$$B=(b_{r,l})\sim\left(\frac{1}{(l-r)\,l^2z}+\frac{1}{(l^2-r^2)\,z}\right)(r\to\infty,\ l\to\infty) \tag{2.14}$$

All the estimates of the elements of the matrices of (2. 13) and (2, 14) are given correctly up to the factors of the form $c\ln l$ and $c\ln r$.

In what follows we shall use the following result, the proof of which is omitted for the sake of brevity . If $X=(x_l)\in c_1$, then

$$\sum_{l=1}^{\infty}{}'\frac{|x_l|}{|l-r|}=y_r,\qquad Y=(y_r)\in m\qquad (r=1,2,\ldots)$$

$$Y_1=(y_r r^{-\gamma}\ln r)\in c_1\qquad (0<\gamma<1) \tag{2.15}$$

The prime of the symbol Σ denotes that the term corresponding to $r=l$ is dropped .

To ensure that, for instance, (2) is satisfied in the case of (2. 14), it is sufficient to prove the convergence of the series

$$\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}\sum_{k=1}^{\infty}|\tau_{l,r}b_{r,k}|\,l^{-1}$$

This series is majorized by the series cJ (c = const)

$$J=\frac{1}{|z|}\sum_{l=1}^{\infty}{}'\sum_{r=1}^{\infty}{}'\sum_{k=1}^{\infty}{}'\frac{1}{l}\,\frac{l^\gamma}{r^\gamma|r-l|}\,\frac{1}{|k-r|}\left(\frac{1}{k^2}+\frac{1}{k+r}\right)$$

Then, on the basis of (2. 15), we obtain

$$J \leqslant \frac{c_2}{|z|}\left(\sum_{k=1}^{\infty}{}' \sum_{r=1}^{\infty}{}' \frac{1}{|k-r|\, r^{\gamma} k^2} + \sum_{r=1}^{\infty}{}' \sum_{k=1}^{\infty}{}' \frac{1}{|k-r|\, r^{\gamma}(k+r)}\right) <$$

$$< \frac{1}{|z|} c_2 \left(c_3 \sum_{k=1}^{\infty} \frac{1}{k^2} + c_4 \sum_{r=1}^{\infty} \frac{1}{r^{\gamma}(1+r)^{1-\gamma/2}}\right) < \infty$$

$$c_2 = \sup_r \sum_{l=1}^{\infty}{}' \frac{1}{l^{-\gamma}|r-l|}, \quad c_3 = \sup_k \sum_{r=1}^{\infty} \frac{1}{r^{\gamma}|k-r|}, \quad c_4 = \sup_k \sum_{r=1}^{\infty} \frac{1}{|k-r|(k+r)^{\gamma/2}}$$

Now it has also been proved that condition (4) is satisfied. It is clear that we can take any number as close to zero as desired as β (see (2) to (5)).

In a completely analogous way we can verify that all the conditions (2) to (5) indicated in Section 1 are satisfied.

Thus, the result of Section 1 is applicable to the infinite systems (2.10) and (2.12). Their solutions can be written as the limit for $n \to \infty$ of the recurrence process (1.5).

Requiring now that the argument z in the solutions of the systems which have been obtained vary on the positive part of the real axis, and constructing Expressions (2.2) and (2.7), we find the solutions of the integral equations (2.1) and (2.4). It is now necessary to ensure that the solutions x_k which have been obtained are bounded as functions of a for $a > 0$, i.e. that there is no value $a_1 > 0$ such that some $x_k(a) \to \infty$ as $a \to a_1$. Or, in other words, we must verify that the real axis does not intersect the set determined by Equation (1.12).

We shall restrict ourselves to the case of Equation (2.1), since for Equation (2.4) everything is carried out in an analogous manner.

We shall assume that a uniqueness theorem holds for Equation (2.1), i.e. that from Equation

$$\int_{-a}^{a} k(x-\xi)\, q(\xi)\, d\xi = 0 \qquad (|x| \leqslant a,\ 0 < a < \infty) \tag{2.16}$$

it follows that $q(\xi) \equiv 0,\ |\xi| \leqslant a$.

Let us assume that (2.2) is the solution of Equation (2.1). Then

$$K^{-1}(\eta) \int_{-a}^{a} k(x-\xi)\, e^{in\xi}\, d\xi + \sum_{l=1}^{\infty} \Big[y_l^{+}(a, \eta) \int_{-a}^{a} k(x-\xi) \exp iz_l(a+\xi)\, d\xi +$$

$$+ y_l^{-}(a, \eta) \int_{-a}^{a} k(x-\xi) \exp iz_l(a-\xi)\, d\xi \Big] \equiv \pi e^{i\eta x} \qquad (|x| \leqslant a) \tag{2.17}$$

Taking into account the linear independence of the functions $\exp i\eta\chi$, $\exp iz_k\chi$ ($k = 1, 2, \ldots$) and also the condition (2.16), we can see that all the terms on the left-hand side of the identity (2.17) are linearly independent functions and, therefore, that the terms containing coefficients which might be unbounded can be regarded as mutually independent.

Let us assume that some $y_k^{+}(a, \eta) \to \infty$ for $a \to a_1$. But then from the boundedness of the terms of the series of (2.17) for all a, it follows that

$$\int_{-a_1}^{a_1} k(x-\xi) \exp iz_k(a+\xi)\, d\xi \equiv 0 \qquad (|x| \leqslant a)$$

which contradicts the condition (2.16). Thus, all coefficients of the expansion (2.2) are bounded for $0 < a < \infty$, and it follows from Equations (2.3) that the corresponding $x_k(a)$ are also bounded.

3. Let us examine the case of Equations (2.1) in greater detail. By finding corresponding values of Y^+ and Y^- on the basis of Equations (2.11) and substituting into (2.2), we can write an expression for $q_\eta(\xi)$ in the form

$$q_\eta(x) = K^{-1}(\eta)\, e^{i\eta x} - K_+^{-1}(\eta)\, e^{-i\eta a}\, \psi(i\eta,\, a+x) - K_-^{-1}(\eta)\, e^{i\eta a}\, \psi(-i\eta,\, a-x) + \sum_{k=1}^{\infty} [\sigma_k^+ \psi(-iz_k,\, a+x) + \sigma_k^- \psi(-iz_k,\, a-x)] \tag{3.1}$$

The quantity $\sigma_k^\pm$ is obtained from the expansion

$$x_l^+ \pm x_l^- = \lim_{n\to\infty} (x_l^+(n) \pm x_l^-(n)) = 2c_l(\pm\eta)\, e^{\pm i\eta a} - \sum_{k=1}^{\infty} \frac{2\sigma_k^\pm}{K_+'(-z_l)(z_k+z_l)}$$

$$\psi(\tau, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-izt}\, dz}{K_+(z)(z - i\tau)}$$

$$\sigma_k \sim O(\exp 2aiz_k) \qquad (k \to \infty) \tag{3.2}$$

Considering that $\psi(\tau, t)$ is the solution of a Wiener-Hopf integral equation of the following form:

$$\int_0^\infty k(t-\xi)\, \psi(\tau, \xi)\, d\xi = \pi e^{\tau t} + \frac{1}{K_+(i\tau)} \int_0^t k(t-\xi)\, e^{\tau\xi}\, d\xi \qquad (0 \leqslant t < \infty) \tag{3.3}$$

we can formulate the result obtained in (3.1) as follows. The solution of the integral equation (1.1) can be expanded into a series (of the form (3 1)) of solutions of the integral equations (3.3) with $\tau = -iz_k$. The coefficients of the expansion are obtained in accordance with Equation (3.2).

The set of the first three terms of the expansion (3.1) provides the zeroth term of the asymptotic solution of Equation (1.1) for $a \to \infty$, which was first obtained in [4], and somewhat later in [5]. The additional series is a regular expansion of the remainder term. As is apparent from (3.2), this series converges like a geometric progression, and its convergence deteriorates only when $a \to 0$.

A representation which is analogous to (3.1) can also be given for the solution (2.7) of Equation (2.4). However, in view of its complicated character, this formula is not given (it can be obtained on the basis of the relations (2.7), (2.12) and (1.5)). We mention that the corresponding σ_k, unlike (3.2), are now of the order $O(\frac{1}{2} z_k a)$.

In the practical use of the results which have been found, it is important to be aware of the fact that if the process (1.5) is terminated at the nth step, then in Expression (3.1) a partial sum consisting of the first n terms is obtained instead of the whole series. This makes it possible to control the accuracy of the results obtained, by investigating the orders of the successive terms which are given.

We note that if the approximate solutions q_η and q of equations (a) and (b) found by the method indicated above have a certain accuracy for $a = a^*$, then this accuracy is not reduced for values $a \geq a^*$.

This fact permits us to construct a combination of solutions with the required accuracy, consisting of solutions obtained by other methods for $0 < a \leq a^*$ and the solution obtained by the method proposed above. It is sufficient to construct q_η and q with the required accuracy for $a = a^*$.

As an example, let us consider the integral equation of the plane contact problem of the indentation of a perfectly rigid flat die into an elastic strip which rests without friction on a rigid base [6]. There is no friction between the die and the layer.

The integral equation can be represented in the form of (2.1). We introduce the notation

$$a = a_1 / h, \quad x = y / h, \quad \xi = \eta / h, \quad q_0(\eta) = q^*_0(\eta / h)$$

where a_1 is half the length of the line of contact, h is the thickness of the layer, $q_0^*(\xi)$ is the contact stress, and $\pi\Delta / h$ is the right side of the integral equation, with

$$K(u) = \frac{L(u)}{u} = \frac{\sinh^2 u}{u(\sinh 2u + 2u)}, \qquad \Delta = \frac{E}{2(1-\sigma^2)} \tag{3.4}$$

By factorizing $K(u)$ into an infinite product, we can write the solution of the integral equation in accordance with (3.1) (*). However, it is difficult to obtain formulas which are suitable for practical use in this way.

It is, therefore, proposed to carry out a twofold approximation, namely

$$K(u) \approx \frac{\sqrt{u^2 + B^2}\, p_1(u)}{(u^2 + c^2)\, r_1(u)} \tag{3.5}$$

$$K(u) \approx \frac{\Gamma(1 + g/\beta - iu/\beta)\,\Gamma(1 + g/\beta + iu/\beta)\, p_2(u)}{\Gamma(1 + b/\beta - iu/\beta)\,\Gamma(1 + b/\beta + iu/\beta)\, r_2(u)} \qquad (b - g = \beta\gamma) \tag{3.6}$$

where $\Gamma(t)$ is the gamma function, and p_k and r_k are even polynomials of equal degrees. The use of the approximation (3.5) permits us to represent the function $\psi(\tau, t)$ in the form [7] (and by the same token to separate out the characteristic singularity for $t = 0$)

$$\psi(\tau, t) = K_+^{-1}(i\tau)\, e^{\tau t} \operatorname{erf} c \sqrt{(B + \tau)\, t} - (\pi t)^{-0.5} e^{-Bt} - \sum_{k=1}^{n} c_k(\tau) \operatorname{erf} \sqrt{(B - i\xi_k)\, t}$$

$$K_+(u) = \frac{\sqrt{B - iu}\, p_1^+}{(c - iu)\, r_1^+}, \qquad c_k(\tau) = \left.\frac{(c - iu)\, r_1^+(u)}{\sqrt{B - iu}\, p_1^{+1}(u)}\right|_{u=\xi_k} \tag{3.7}$$

where ξ_k are the roots of the polynomial p_1 which lie in the upper half-plane.

The approximation (3.6) allows us to obtain σ_k in a rather simple form, as for example

$$\sigma_1^+(\eta) = \frac{c_1(-\eta)\, e^{i\eta a} - \varepsilon_{1,1}\, c_1(\eta)\, e^{-i\eta a}}{1 - \varepsilon_{1,1}^2} K_+(z_1) \exp 2aiz_1 \tag{3.8}$$

For the approximation (3.5), the expressions which occur in (3.7) are easily computed.

The possibility of introducing the approximations (3.5) and (3.6) follows from the fact that the solution $q_\eta(\xi)$ in (3.11) can be written in the form of integrals along the real axis containing the functions $K_+(u)$ and $K_-(u)$. Consequently, the solution does not change greatly if these functions are replaced by their approximations.

In the case of problem (3.4), the approximating functions corresponding to (3.5) and (3.6) have the form (p_k and r_k are of fourth degree)

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*) The presence of multiple roots in the function $K(u)$ is obviously unimportant. It is possible, for instance, to pass to the limit in the equations, considering that two neighboring zeros coalesce.

$$\frac{\sqrt{u^2+1}\,(u^4+12.469\,u^2+24.692)}{(u^2+2)\,(u^4+10.115\,u^2+24.692)}\,, \qquad \frac{\tanh 0.5\,u}{u}\,\frac{u^4+5.1689\,u^2+4.6757}{u^4+4.3378u^2+4.6757} \tag{3.9}$$

$$\xi_1 = 1.5713i,\ \xi_2 = 3.1623i,\ z_1 = 1.0812i,\ z_2 = 2i,\ z_{n+2} = 2\pi n i$$

The error of this approximation does not exceed 2.5% .

We shall now attempt to construct the solution of the problem correct to three significant figures for the values $0.25 \leq a < \infty$, using the approximation in (3.9). In order to do this, it proves to be sufficient to take two terms of the series and the principal component $\psi_1(x)$ of the third term, since all the succeeding terms do not affect the third significant figure for $a = 0.25$.

The solution may be written in the form

$$q_0(x) = \Delta h^{-1} \{2 - \sqrt{2}\,[\psi(0, a+x) + \psi(0, a-x)] + \sigma_1\,[\psi(1.0812, a+x) + \psi(1.0812, a-x)] + \sigma_2\,[\psi(2, a+x) + \psi(2, a-x)] + \psi_1(x) + o\,(e^{-12.566a})$$

$$\sigma_1 = -\left[\frac{x_1(0)}{1+\varepsilon_{1,1}}\left(1+\frac{\varepsilon_{12}\varepsilon_{21}}{\Delta_1}\right) - \frac{x_2(0)\,\varepsilon_{12}}{\Delta_1}\right] 0.29633\,e^{-2.1623a}$$

$$\sigma_2 = -\left[\frac{x_2(0)}{1-\varepsilon_{2,2}}\left(1+\frac{\varepsilon_{12}\varepsilon_{21}}{\Delta_1}\right) - \frac{x_1(0)\,\varepsilon_{21}}{\Delta_1}\right] 0.26088\,e^{-4a} \tag{3.10}$$

In Equations (3.10)

$$x_k(0) = x_k^+(0),\ \varepsilon_{i,k} = \varepsilon_{i,k}^+(0) \qquad \Delta_1 = 1 + \varepsilon_{1,1} + \varepsilon_{2,2} + \varepsilon_{1,1}\varepsilon_{2,2}$$

$$\varepsilon_{1,1} = 0.046420\,e^{-2.1623a}, \qquad \varepsilon_{2,2} = -0.025385\,e^{-4a}$$

$$\varepsilon_{1,2} = 0.028681\,e^{-4a}, \qquad \varepsilon_{2,1} = -0.037433\,e^{-2.1623a}$$

$$\psi_1(x) = \frac{x_3(0)\,0.35836\,e^{-12.566a}}{(1+\varepsilon_{3,3})}\left(\frac{e^{-(a+x)}}{\sqrt{\pi(a+x)}} + \frac{e^{-(a-x)}}{\sqrt{\pi(a-x)}}\right)$$

For comparison, we give the results of calculations for $q^* = a_1 q_0(x)/\Delta$ according to (3.10) together with the corresponding result obtained by the method of large $\lambda = h/a_1$ due to Aleksandrov [6].

$x =$	0	0.2	0.4	0.6	0.8	0.95	
$q^* =$	0.586	0.590	0.632	0.717	0.934	1.785	0.551
$q_{[6]} =$	0.597	0.608	0,645	0.730	0.956	1.809	0.562

The last column of the table contains values of

$$\lim_{x\to a} \sqrt{a^2-x^2}\,a_1 q_0(x)\,/\,\Delta$$

Apparently, the difference in values of the solutions which are seen from the table arise as a result of use of the approximation of (3.9). Numerical analysis indicates that the bigger a is, the more stable is the solution as regards approximations. For instance, for $a = 0.5$, the deviation of the solution (3.10) from the corresponding solution obtained by the method of large λ is reduced to under 1.5%.

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